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LETTER TO THE EDITOR

Remarks on stochastic resonances

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Abstract. A formalism for discussing the notion of stochastic resonance is outlined and applied to a simple two-state model.

Recently, there has been interest in the cooperative effect of an *internal* random mechanism and an *external* time-periodic driving mechanism on certain dynamical systems (Benzi *et al* 1981a, b). The contention is that the two mechanisms together can enhance the power spectrum of some dynamical variable $x(t)$ over a frequency interval $\Delta\omega$, in which case one speaks of a *stochastic resonance.*

Although the notion of stochastic resonance is not well defined by the above (nor do we attempt a precise definition here), it should be distinguished from the case of a dynamical system with a resonant frequency ω_0, driven nearly at this frequency, and perturbed by a small random component. For the stochastic resonance, only the interplay between the random mechanism and periodic driving mechanism is at issue, and no underlying 'natural' frequency of the dynamical system plays any role.

The authors, Benzi, Parisi *et al*, consider a one-dimensional diffusion in a symmetric double-well potential with linear time-dependent potential as external driving mechanism. The analysis of this system is by no means trivial. For example, expected escape times from one well to the other will be solutions to *two*-dimensional partial differential equations (the second dimension arising from the time dependence in the problem). Nevertheless, after some approximations, the authors do present evidence of a stochastic resonance in the power spectrum for the particle position $y(t)$.

Here, we discuss an even simpler model, a two-state model, with states labelled (+) and (−), in which the state changes at an exponentially distributed time with rate modified by a time-periodic factor. (Our formalism accommodates other finite- and infinite-state models. With suitable modifications, the formalism will accommodate diffusions.) As dynamical variables, we consider $N(t)$, the number of times the system jumps from the (−) state to the (+) state up to time t. In addition to giving integral expressions for the moments of $N(t)$, we explicitly compute the mean and variance of $N(t)$ for large t, as a function of the driving frequency ω. We show that there is an optimal frequency at which $N(t)$ is, in a sense made precise below, most periodic-like in its behaviour, indicating an enhancement of the power spectrum for $N(t)$ about

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this frequency. Finally we conclude with an alternative approach, analysing the sojourn time in the (+) and (−) states.

We now consider the two-state model. Let

$$\mu_\eta(t)=\begin{pmatrix}\mu_\eta^+(t)\\ \mu_\eta^-(t)\end{pmatrix}, \qquad \mu_\eta(0)=\begin{pmatrix}1\\0\end{pmatrix}, \tag{1}$$

be the solution to the differential equation

$$\mathrm{d}\mu_\eta(t)/\mathrm{d}t = H_\eta(t)\mu_\eta(t) \tag{2}$$

with

$$H_\eta(t)=\begin{pmatrix}-h_+(t) & \eta h_-(t)\\ h_+(t) & -h_-(t)\end{pmatrix} \tag{3}$$

and $h_+(t)$, $h_-(t)$ non-negative periodic functions describing the rates at which the system changes its state from (+) to (−) and (−) to (+) respectively. The boundary conditions for $\mu_\eta(t)$ are chosen for the sake of definiteness only, and are irrelevant to the results below. The parameter η is introduced to develop a generating function for the process to be studied. For $\eta=1$, $H_\eta(t)$ is a stochastic generator; hence $\mu(t)\equiv\mu_1(t)$ is a probability vector with $\mu^+(t)$ ($\mu^-(t)$) the probability that the (+) ((−)) state is occupied. Associated with $\mu(t)$ is a stochastic process $x(t)$ with $x(t)=\pm1$ according to whether the (+) or (−) state is occupied at time t.

As before, we let $N(t)$ be the random process equal to the number of transitions executed from (−) to (+), up to time t. Set

$$m_\eta(t)\equiv\mu_\eta^+(t)+\mu_\eta^-(t) \tag{4}$$

and consider the power series expansion for $m_\eta(t)$ in η,

$$m_\eta(t)=\sum_n m^n(t)\eta^n=\lim_{N\to\infty}(1,1)\prod_{j=1}^{N}\left(1+\frac{1}{N}H_\eta\left(\frac{jt}{N}\right)\right)\begin{pmatrix}1\\0\end{pmatrix}. \tag{5}$$

The coefficients $\{m^n(t)\}$ are non-negative and sum to unity since $m_1(t)=1$. Moreover $m^n(t)$ is homogeneous of degree n in the upper right-hand entry of H_η, i.e. h_-, hence $m^n(t)$ is the probability that $N(t)=n$. Thus

$$m_\eta(t)=\mathrm{Exp}(\eta^{N(t)}) \tag{6}$$

is the moment generating function for $N(t)$, where Exp denotes expectation. (If $\eta=\mathrm{e}^{-\alpha}$, $\mathrm{Exp}(\mathrm{e}^{-\alpha N(t)})$ is simply the Laplace transform of the measure for $N(t)$.) In particular, we have

$$\mathrm{Exp}\,N^n(t)=(\eta\partial/\partial\eta)^n m_\eta(t)|_{\eta=1}. \tag{7}$$

Manipulation of equations (1)–(4) leads to a second-order ordinary differential equation for $m_\eta(t)$,

$$\frac{\mathrm{d}^2m_\eta(t)}{\mathrm{d}t^2}+(h_+(t)+h_-(t)-h_-'(t)/h_-(t))\frac{\mathrm{d}m_\eta(t)}{\mathrm{d}t}-(\eta-1)h_+(t)h_-(t)m_\eta(t)=0 \tag{8}$$

with

$$m_\eta(0)=1, \qquad \mathrm{d}m_\eta(0)/\mathrm{d}t=0, \tag{9}$$

which can be converted to an integral equation,

$$m_\eta(t)=1+(\eta-1)Km_\eta(t) \tag{10}$$

where K is the integral operator defined by

$$Km(t)=\int_0^t\left[\int_s^t h_-(s')\exp\left(-\int_s^{s'}(h_+(u)+h_-(u))\,\mathrm{d}u\right)\mathrm{d}s'\right]h_+(s)m(s)\,\mathrm{d}s. \tag{11}$$

To compute moments of $N(t)$, equation (7), we note that we need only compute $m_\eta(t)$ for η near 1. This suggests the Neumann series solution to equation (10),

$$m_\eta(t)=\sum_{l=0}^{\infty}(\eta-1)^l K^l \mathbb{1}(t) \tag{12}$$

with $\mathbb{1}$ the function which is identically 1. Combining this with equation (7), we obtain

$$\operatorname{Exp}[N^n(t)]=(\eta\partial/\partial\eta)^n\sum_{l=0}^{n}(\eta-1)^l K^l\mathbb{1}(t)|_{\eta=1}. \tag{13}$$

The simple model described above is a special case of a more general formalism. Let X be a compact Hausdorff space (e.g. a finite set, a bounded closed set in $\mathbb{R}^n$, $\mathbb{R}^n$ with a point at ∞, etc) and let $C(X)$ be the space of continuous functions on X. Let $h_0^\dagger$ be a stochastic semigroup generator on $C(X)$ and let $h_+(t)$, $h_-(t)$ be operator-valued functions on the real line which, for each t, map positive measures on X to positive measures on X. Consider the evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mu_\eta^+(t)\\ \mu_\eta^-(t)\end{pmatrix}=\begin{pmatrix}h_0-h_+(t) & \eta h_-(t)\\ h_+(t) & h_0-h_-(t)\end{pmatrix}\begin{pmatrix}\mu_\eta^+(t)\\ \mu_\eta^-(t)\end{pmatrix} \tag{14}$$

for measures on $X\oplus X$. (Here, the state space is $X\oplus X$.) Setting

$$m_\eta(t)=\mu_\eta^+(t)+\mu_\eta^-(t), \qquad \mu_\eta^+(0)=\mu_0, \qquad \mu_\eta^-(0)=0, \tag{15}$$

we obtain the integral equation

$$m_\eta(t)=m_1(t)+(\eta-1)Km_\eta(t) \tag{16}$$

with

$$m_1(t)=\exp(th_0)\mu_0 \tag{17}$$

and K the integral operator defined by

$$Km(t)=\int_0^t \exp[(t-s)h_0]h_-(s)\int_0^s\left[T\exp\left(\int_{s'}^s(h_0-h_+-h_-)\,\mathrm{d}u\right)\right]h_+(s')m(s')\,\mathrm{d}s'\,\mathrm{d}s. \tag{18}$$

(Here $U(t)=T\exp(\int_{t_0}^t k(u)\,\mathrm{d}u)$ denotes the time-ordered exponential and is the solution to the differential equation

$$\mathrm{d}U(t)/\mathrm{d}t=k(t)U(t)$$

with $U(t_0)=\mathbb{1}$.) If $N(t)$ is the number of transitions from the second copy of X to the first copy of X up to time t, then as in equation (6)

$$\int_X m_\eta(t)=\operatorname{Exp}\eta^{N(t)} \tag{19}$$

where $\int_X$ denotes integration over X. The moments of $N(t)$ are given by

$$\operatorname{Exp}N^n(t)=\left(\eta\frac{\partial}{\partial\eta}\right)^n\int_X\sum_{l=0}^{n}(\eta-1)^l K^l m_1(t)|_{\eta=1}. \tag{20}$$

Returning to the two-state model, we consider the particular case

$$H_\eta(t)=\begin{pmatrix}-(a+\varepsilon\cos\omega t) & \eta(a-\varepsilon\cos\omega t)\\ (a+\varepsilon\cos\omega t) & -(a-\varepsilon\cos\omega t)\end{pmatrix}. \tag{21}$$

One finds, for example, that

$$\operatorname{Exp} N(t)=\tfrac{1}{2}at[1-2\varepsilon^2/(4a^2+\omega^2)]+\mathrm{O}(1), \tag{22}$$

$$\operatorname{var} N(t)=\operatorname{Exp}[N^2(t)]-(\operatorname{Exp} N(t))^2=\frac{at}{4}\left(1-\frac{8\varepsilon^2\omega^2}{4a^2+\omega^2}-\frac{2\varepsilon^4(12a^2-\omega^2)}{(4a^2+\omega^2)^3}\right)+\mathrm{O}(1), \tag{23}$$

where the O(1) terms are of order 1 in t. As one expects, both the mean and variance are proportional to t plus lower-order corrections.

As a measure of the quality of the periodicity of $N(t)$ we consider the quantity

$$Q\left(\frac{\varepsilon}{a},\frac{\omega}{a}\right)=\lim_{t\to\infty}\frac{at\operatorname{var} N(t)}{(\operatorname{Exp} N(t))^2}. \tag{24}$$

If Q is small we infer that the fluctuations of $N(t)$, as estimated by var $N(t)$, are small in comparison with $N^2(t)$ itself; that is, the transitions from (−) to (+) are approximately periodic and there is a peak in the power spectrum of $N(t)$. A graph of Q is given in figure 1 for various values of ε/a. Note that Q is identically 1 for $\varepsilon/a=0$. When $\varepsilon/a=1$ the dip is most pronounced, $Q_{\min}$ equalling about 0.791.

The dip we see here is rather mild. Benzi *et al* (1981a) observe a more pronounced dip in their numerical simulations. We believe that a more complicated barrier structure (using e.g. a three-state model) would produce a sharper resonance, because at high frequency, ± jumps would become less probable than in our model.

The reader might ask whether the dynamical variable $x(t)$ exhibits stochastic resonance. (Recall that $x(t)=\pm1$ according to whether the system is in the (+) or (−) state at time t.) Let

$$\hat{x}_T(\omega')=\frac{1}{T^{1/2}}\int_0^T \exp(\mathrm{i}\omega' s)x(s)\,\mathrm{d}s. \tag{25}$$

Then the power spectrum for $x(t)$ corresponding to the example equation (21) is given by

$$\operatorname{Exp}[\hat{x}_T(\omega')]=\frac{-\varepsilon T^{1/2}}{2a-\mathrm{i}\omega}\Delta(\omega-\omega')+\mathrm{O}(T^{-1/2}), \tag{26}$$

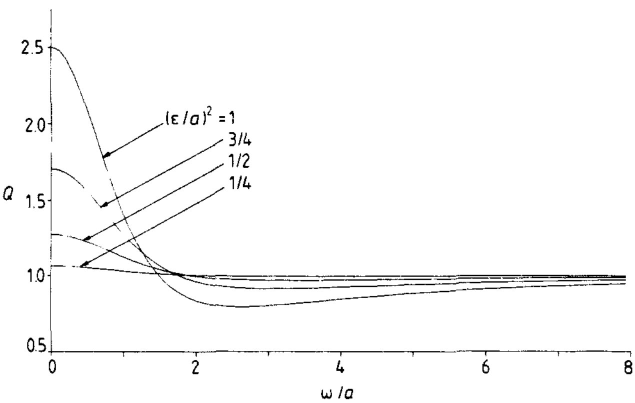

Figure 1. Graph of Q as a function of ω/a for $(\varepsilon/a)^2=\frac{1}{4},\frac{1}{2},\frac{3}{4},1$.

$$\mathrm{Exp}|\hat{x}_T(\omega')|^2 - |\mathrm{Exp}[\hat{x}_T(\omega')]|^2 = \frac{4a}{4a^2+\omega'^2}\left(1-\frac{2\varepsilon^2}{4a^2+\omega^2}\right)+\mathrm{O}\left(\frac{1}{T}\right), \qquad (27)$$

where $\Delta(\omega')=0$, $\omega'\neq 0$, $\Delta(\omega')=1$, $\omega'=0$. These relations apparently do not exhibit a resonance, indicating that the choice of dynamical variable is important, at least in this simple model.

Finally, we note an alternative but closely related approach to stochastic resonances. Rather than considering the *frequency* of jumps up to time t, $N(t)/t$, we consider instead the *periods* of sojourn in the (+) or (−) states, up to time t. We sketch this point of view in the abstract setting of equation (14), with $\eta=1$.

Corresponding to the evolution equation (14) is a process $x(t, x_0, s)$, the position at time t in $X\oplus X$, given the position x_0 at time s. Because of the time dependence of the generator in this equation (the generator is assumed to be periodic with period $2\pi/\omega$) the process is certainly not stationary. But the process

$$y(\sigma, x_0, t_0) = (x, t) \equiv (x(\sigma+t_0), x_0, t_0), \mathrm{res}(t_0+\sigma) \qquad (28)$$

defined on $(X\oplus X)\times[0, 2\pi/\omega)$ ($\mathrm{res}(t_0+\sigma)$ denotes the residue of $(t_0+\sigma)\bmod(2\pi/\omega)$) is stationary in the augmented time σ. With the stationary process in hand, it is then straightforward to analyse the sojourn times (cf Eckmann *et al* 1981), which we now define.

Let $\tau_0=0$ and set inductively $\tau_j=\inf(\sigma>\tau_{j-1}|y$ jumps from a (−) to (+) or (+) to (−) state at time σ). Thus τ_j is simply the (σ)-time of the jth jump from (+) to (−) or (−) to (+) states. The jth sojourn time is then $\tau_j-\tau_{j-1}$; the average sojourn time and higher moments of it are given by

$$\mathbb{T}(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(\tau_j-\tau_{j-1})^n, \qquad n=1,2,\ldots. \qquad (29)$$

Under mild hypotheses on the generator of equation (14) these limits exist and are equal to their expected values almost surely,

$$\mathbb{T}(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\mathrm{Exp}(\tau_j-\tau_{j-1})^n. \qquad (30)$$

In order to arrive at an analytical expression for $\mathbb{T}(n)$ we note that

$$\mathrm{Exp}(\tau_j-\tau_{j-1})^n = \mathrm{Exp}[\mathrm{Exp}((\tau_j-\tau_{j-1})^n|y(\tau_{j-1}))] \qquad (31)$$

(we assume that $y(\tau_{j-1})=\lim_{\sigma\downarrow\tau_{j-1}} y(\sigma)$). Setting

$$T(n, y) = \mathrm{Exp}[(\tau_1-\tau_0)^n|y(\tau_0)=y], \qquad (32)$$

we find

$$\mathbb{T}(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}T[n, y(\tau_{j-1})] = \int_{(X\oplus X)\times[0,2\pi/\omega)} T(n, y)\,\mathrm{d}\rho(y) \qquad (33)$$

where $\rho(y)$ is the invariant measure for the process $y(\tau_0)$, $y(\tau_1)$, $y(\tau_2), \ldots$.

In terms of the generator coefficients of equation (14), the conditional expectations can be written as

$$T(n, y) = -\int_0^\infty t^n\,\mathrm{d}P_y(t) \qquad (34)$$

with $P_y(t)$ the distribution given by

$$P_y(t) = \text{Prob}[(\tau_1 - \tau_0) > t | y(\tau_0) = y] = \left(T \exp \int_s^{t+s} (h_0(u) - h_\pm(u))\, \mathrm{d}u\right)^\dagger \mathbb{1}(x) \tag{35}$$

where $y = (x, s)$, the $\pm$ is chosen depending on whether x is in the (+) or (−) states, and $\mathbb{1}$ is the function which is identically 1 on X. (Again, the time-ordered exponential is required.) The density ρ is the (normalised) solution to the integral equation

$$\mathrm{d}\rho(y) = \int \mathrm{d}K(y, y')\, \mathrm{d}\rho(y') \tag{36}$$

with $\mathrm{d}K(y, y')$ the probability density that a sojourn begun at y' belonging to the (+) ((−)) states ends by a jump to y' belonging to the (−) ((+)) states. This density is given implicitly by the solution to a Dirichlet problem (cf Feller 1966). If f_0 is a continuous function on the (−) ((+)) states,

$$f(y') \equiv \int f_0(y)\, \mathrm{d}K(y, y') \tag{37}$$

defined on the (+) ((−)) states satisfies the differential equation

$$\partial f/\partial t + (h_0^\dagger - h_\pm^\dagger(t))f = -h_\pm^\dagger(t) f_0. \tag{38}$$

This equation can be integrated to deduce $\mathrm{d}K(y, y')$:

$$\mathrm{d}K(y, y') = \sum_{\substack{n \\ t+2\pi n/\omega \geqslant t'}} \left[\left(T \exp \int_{t'}^{t+2\pi n/\omega} (h_0 - h_\pm(u))\, \mathrm{d}u\right)^\dagger h_\pm^\dagger(t)\right](x, x')\, \mathrm{d}t \tag{39}$$

with $y = (x, t)$, $y' = (x', t')$.

One can then study the moments $\mathbb{T}(n)$ given by equation (33) as a function of ω, the driving frequency in $h_\pm(t)$. For the two-state model, even with the choice $h_\pm(t) = (a \pm \varepsilon \cos \omega t)$ defined by equation (21), the calculation of ρ is not elementary (although it, along with $T(n, y)$, can be determined in low-order perturbation theory in ε) and so we do not pursue this program here. We do emphasise that both the function $T(n, y)$ and the density ρ influence the behaviour of $\mathbb{T}(n)$ as a function of ω; a resonance will be a collective effect arising from the dependence of $T(n, y)$ and ρ on ω.

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